Consistency of the particle-particle random-phase approximation and renormalization

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The consistency condition is tested within the particle-particle random-phase approximation (RPA),
renormalized RPA (RRPA), and the self-consistent RPA (SCRPA) making use of the Richardson model of
pairing. The two-particle separation energy is calculated in two ways, namely, as the energy of the first addition
mode, which adds two particles to a core with N particles, and as the energy of the first removal mode, which
removes two particles from the $N + 2$ particle system to get back to the same N-particle core. The corresponding
transitions generated by the pairing operators are also calculated. It is found that the results obtained in these two
ways of calculations are close to each other only at large values of particle number N and/or small interaction
strength. At $N \leqslant 10$ for a given value of the interaction strength, the discrepancy between the results obtained in
two ways of calculations within the SCRPA is much smaller than those given by the RPA and RRPA.

I. INTRODUCTION

The random-phase approximation (RPA) has been widely
used in the theoretical study of nuclei within the valley of
β-stability. The success of the RPA is mainly based on the
use of the quasiboson approximation (QBA), which considers
fermion pairs as boson operators, just neglecting the Pauli
principle between them. As a result, a set of linear equations,
called the RPA equations, was derived, which reveals the
physics of collective excitations generated by the RPA boson-
like modes. The simplicity of the RPA equations allows a
feasible treatment of a number of many-body problems, which
would be computationally intractable otherwise. However, this
approach suffers a drawback: It breaks down at a certain
critical value of the interaction's parameter, where the RPA
yields imaginary solutions. The reason of this well-known
RPA instability is the violation of Pauli principle within the
QBA.

In β-stable medium and heavy nuclei, the QBA is a good
approximation, and the RPA is a very powerful tool for the
description of several important quantities such as the ground-
state and excited-state energies, electro-magnetic transition
probabilities and their distribution, transition densities, etc.
The first-order diagram expansion beyond the mean field
treated within the RPA includes significant effects of two-body
correlations beyond the mean-field. However, with reducing
the particle number, the concept of collective excitations,
which are described by the RPA modes, becomes less and
less firm. The ground-state correlations (GSC) which are left
beyond the RPA become stronger in light systems. This feature
makes the validity of the QBA, and therefore of the RPA itself,
questionable in the systems with small particle numbers.

Several approaches were developed to take into account
the GSC beyond RPA in a simple way such as to restore

the Pauli principle among the fermion pairs, from which the
RPA operators are constructed. The popular one, known as
the renormalized RPA (RRPA) [1–4], includes the expectation
value over the ground state of the diagonal elements of the
commutator between two fermion-pair operators, neglected in
the QBA. In this way the Pauli principle is approximately taken
care of. The inclusion of GSC beyond RPA within the RRPA
eventually renormalizes the interaction in such a way that the
collapse of RPA is avoided, and the RRPA has solutions at
any value of the interaction's parameter. This scheme has been
tested in several exactly solvable schematic models such as the
Lipkin model for particle-hole (ph) RRPA [4], the SO(8) model
for the renormalized quasiparticle RPA [5], and the Richardson
model for the particle-particle (pp) RRPA and self-consistent
RPA (SCRPA) [6,7]. The results of these studies showed that
the difference between the energies of the first excited state
given by the (pp) RPA, (pp) RRPA, and SCRPA increases
noticeably with increasing the interaction parameter and/or
decreasing the particle number. The SCRPA solutions are the
closest ones to the exact solutions. These results raise a serious
question of the applicability of the RPA to the calculations
for realistic light-neutron rich nuclei. As a matter of fact, a
close agreement between the RPA results and the experimental
data for light neutron-rich nuclei can be considered really
reasonable if the corrections due to GSC beyond RPA are
negligible, which justify the validity of the QBA [8]. In the
opposite case such agreement may be even illusory.

This concern is the motivation of the present work, whose
goal is to test the consistency of the (pp) RPA and its
renormalizations, the (pp) RRPA and SCRPA, at different
particle numbers as the interaction strength varies. The reason
of choosing the (pp) RPA comes from one of its major
merits, which allows the straightforward calculation of an
important quantity in the study of unstable nuclei, namely the
two-particle separation energy. Indeed, the (pp) RPA uses the
addition and removal phonon operators to create the ground
states and excited states of the systems with $N + 2$ and $N - 2$
particles, respectively, from the ground state of the N-particle

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system. Therefore the two-particle separation energy of the $(N + 2)$-particle system can be calculated either as the energy of the lowest excited state generated by the addition mode, which adds two particles to the core with N particles, or as that generated by the removal mode, which removes two particles from the core with $N + 2$ particles. A consistent theory should give identical results in either way of calculation. By analyzing the results obtained in two ways of calculations within all three approaches, namely, the RPA, RRPA, and SCRPA in their application to an exactly solvable schematic model, namely the Richardson model [9], the present work may shed light on the consistency of these approaches, in particular, in their application to systems with small particle numbers (light systems).

The paper is organized as follows. The brief outline of the (pp) RPA, RRPA, and SCRPA applied to the Richardson model is presented in Sec. II. As the present paper deals only with (pp) RPA and its renormalization, the prefix (pp) will be omitted hereafter. The results of numerical calculations are analyzed in Sec. III. The paper is summarized in the last section, where conclusions are drawn.

II. RPA, RRPA AND SCRPA WITHIN THE RICHARDSON MODEL

A. The Richardson model

The Richardson model, considered in the present paper, consists of Ω doubly-fold equidistant levels, which interact via a pairing force with parameter G. The model Hamiltonian is given as [6,7]

$$H = \sum_{i=1}^{\Omega} (\epsilon_i - \lambda) N_i - G \sum_{i,j=1}^{\Omega} P_i^\dagger P_j, \tag{1}$$

where the particle-number operator N_i and pairing operators $P_i^\dagger$, P_i are defined as

$$N_i = c_i^\dagger c_i + c_{-i}^\dagger c_{-i}, \quad P_i^\dagger = c_i^\dagger c_{-i}^\dagger, \quad P_i = (P_i^\dagger)^\dagger. \tag{2}$$

These operators fulfill the following exact commutation relations:

$$[P_i, P_j^\dagger] = \delta_{ij}(1 - N_i), \tag{3}$$

$$[N_i, P_j^\dagger] = 2\delta_{ij} P_j^\dagger, \quad [N_i, P_j] = -2\delta_{ij} P_j. \tag{4}$$

The single-particle energies of the equidistant levels are defined as $\epsilon_i = i\epsilon$ with i running over all Ω levels. The present paper considers only the ph-symmetric case. This means that, in the absence of interaction ($G = 0$), the lowest $\Omega/2$ levels are filled with $N = \Omega$ particles (two particles on each level). Numerating particle (p) and hole (h) levels from the levels nearest to the Fermi level, the particle and hole energies are equal to $\epsilon_p = \epsilon(\Omega/2 + p)$ and $\epsilon_h = \epsilon(\Omega/2 - h + 1)$, respectively, with p (h) = 1, ..., $\Omega/2$. The Fermi level ϵ_F is defined as

$$\epsilon_F = [\epsilon(\Omega + 1) - G]/2. \tag{5}$$

There are several methods of solving exactly the pairing problem described by Hamiltonian (1), namely, the Richardson method [9], the infinite-dimensional algebras in Ref. [10], and the direct diagonalization of Hamiltonian (1) in the Fock space [11].

B. RPA

The outline of the RPA is given in detail in Ref. [12]. It uses the addition $A_\mu^\dagger$ and removal $R_\lambda^\dagger$ phonon operators to create the ground states and excited states of the systems with $N + 2$ and $N - 2$ particles, respectively, from the ground state of the N-particle system. Applied to the Richardson model, these addition and removal phonon operators are given as

$$A_\mu^\dagger = \sum_p X_p^{(\mu)} Q_p^\dagger - \sum_h Y_h^{(\mu)} Q_h, \tag{6}$$

and

$$R_\lambda^\dagger = \sum_h X_h^{(\lambda)} Q_h^\dagger - \sum_p Y_p^{(\lambda)} Q_p, \tag{7}$$

respectively, where

$$Q_p^\dagger = P_p^\dagger, \quad Q_h = -P_h^\dagger. \tag{8}$$

The RPA assumes the validity of the QBA, which replaces the expectation value of the commutation relations $[Q_i, Q_j^\dagger]$ in the RPA ground state $|N, 0\rangle$ of N-particle system with that obtained in the Hartree-Fock (HF) ground state $|HF\rangle$, namely,

$$\langle N, 0|[Q_i, Q_j^\dagger]|N, 0\rangle \simeq \langle HF|[Q_i, Q_j^\dagger]|HF\rangle = \delta_{ij}, \\ (i, j) = (p, p'), (h, h'), \tag{9}$$

since $\langle HF|N_p|HF\rangle = 0$, and $\langle HF|N_h|HF\rangle = 2$. Using the QBA (9), one can easily verify that the expectation values of the commutation relations for addition and removal operators in the RPA ground state are of bosonic type, i.e.,

$$\langle N, 0|[A_\mu, A_{\mu'}^\dagger]|N, 0\rangle = \delta_{\mu\mu'}, \quad \langle N, 0|[R_\lambda, R_{\lambda'}^\dagger]|N, 0\rangle = \delta_{\lambda\lambda'}, \tag{10}$$

if the X and Y amplitudes obey the following normalization (orthogonality) conditions:

$$\sum_p X_p^{(\mu)} X_p^{(\mu')} - \sum_h Y_h^{(\mu)} Y_h^{(\mu')} = \delta_{\mu\mu'},$$

$$\sum_h X_h^{(\lambda)} X_h^{(\lambda')} - \sum_p Y_p^{(\lambda)} Y_p^{(\lambda')} = \delta_{\lambda\lambda'}, \tag{11}$$

$$\sum_p X_p^{(\mu)} Y_p^{(\lambda)} - \sum_h X_h^{(\lambda)} Y_h^{(\mu)} = 0,$$

while the closure relations

$$\sum_\mu X_p^{(\mu)} X_{p'}^{(\mu)} - \sum_\lambda Y_p^{(\lambda)} Y_{p'}^{(\lambda)} = \delta_{pp'},$$

$$\sum_\lambda X_h^{(\lambda)} X_{h'}^{(\lambda)} - \sum_\mu Y_h^{(\mu)} Y_{h'}^{(\mu)} = \delta_{hh'}, \tag{12}$$

$$\sum_\lambda X_h^{(\lambda)} Y_p^{(\lambda)} - \sum_\mu X_p^{(\mu)} Y_h^{(\mu)} = 0$$

guarantee the following inverse transformation of Eqs. (6) and (7);

$$Q_p^\dagger = \left[\sum_\mu X_p^{(\mu)} A_\mu^\dagger + \sum_\lambda Y_p^{(\lambda)} R_\lambda \right],$$

$$Q_h = \left[\sum_\lambda X_h^{(\lambda)} R_\lambda + \sum_\mu Y_h^{(\mu)} A_\mu^\dagger \right]. \tag{13}$$

The RPA equations are obtained by linearizing the equation of motion with Hamiltonian (1) and operators $A^\dagger$ and $R^\dagger$. The matrix form of the RPA equation for the addition mode is

$$\begin{pmatrix} A & B \\ -B & C \end{pmatrix} \begin{pmatrix} X^{(\mu)} \\ Y^{(\mu)} \end{pmatrix} = E_\mu \begin{pmatrix} X^{(\mu)} \\ Y^{(\mu)} \end{pmatrix}, \tag{14}$$

where the submatrices A, B, and C are found as [6,12]

$$A_{pp'}^{\mathrm{RPA}} = \langle \mathrm{HF} | [Q_p, [H, Q_{p'}^\dagger]] | \mathrm{HF} \rangle$$
$$= 2 \left[\epsilon \left(p - \frac{1}{2} \right) + \frac{G}{2} \right] \delta_{pp'} - G, \tag{15}$$

$$B_{ph}^{\mathrm{RPA}} = -\langle \mathrm{HF} | [Q_p, [H, Q_h]] | \mathrm{HF} \rangle = G, \tag{16}$$

$$C_{hh'}^{\mathrm{RPA}} = -\langle \mathrm{HF} | [Q_h, [H, Q_{h'}^\dagger]] | \mathrm{HF} \rangle$$
$$= -2 \left[\epsilon \left(h - \frac{1}{2} \right) + \frac{G}{2} \right] \delta_{hh'} + G. \tag{17}$$

C. RRPA

The QBA (9) neglects GSC beyond RPA. The latter make the expectation value of N_p deviate from zero and that of N_h deviate from 2. In order to take this effect into account, the RRPA considers the following renormalized addition and removal phonon operators:

$$\mathcal{A}_\mu^\dagger = \sum_p X_p^{(\mu)} \overline{Q}_p^\dagger - \sum_h Y_h^{(\mu)} \overline{Q}_h, \quad \text{and}$$

$$\mathcal{R}_\lambda^\dagger = \sum_h X_h^{(\lambda)} \overline{Q}_h^\dagger - \sum_p Y_p^{(\lambda)} \overline{Q}_p, \tag{18}$$

respectively, with the abbreviation

$$\overline{\mathcal{O}}_i^\dagger = \frac{\mathcal{O}_i^\dagger}{\sqrt{\langle D_i \rangle}}, \quad \overline{\mathcal{O}}_i = \frac{\mathcal{O}_i}{\sqrt{\langle D_i \rangle}},$$
$$\langle D_i \rangle \equiv \langle \widetilde{N, 0} | D_i | \widetilde{N, 0} \rangle, \quad i = p, h, \tag{19}$$

where the correlated ground state $|\widetilde{N, 0}\rangle$ is defined as the vacuum of the operators $\mathcal{A}_\mu$ and $\mathcal{R}_\lambda$, i.e.;

$$\mathcal{A}_\mu | \widetilde{N, 0} \rangle = \mathcal{R}_\lambda | \widetilde{N, 0} \rangle = 0. \tag{20}$$

The GSC factors $\langle D_i \rangle$ are the expectation values of the operators

$$D_p = 1 - N_p, \quad D_h = N_h - 1, \tag{21}$$

in the correlated ground state $|\widetilde{N, 0}\rangle$, Eq. (20). They are introduced in the definition (18) to preserve the ground-state expectation value of the commutation relation (3). Indeed,

using Eq. (3) and definition (8), one obtains the exact commutation relations for operators Q_i and $Q_i^\dagger$ in the form

$$[Q_p, Q_{p'}^\dagger] = \delta_{pp'} D_p, \quad [Q_h, Q_{h'}^\dagger] = \delta_{hh'} D_h. \tag{22}$$

Using this exact relation (22) and the definition (20), one can see that the renormalized addition and removal operators satisfy the boson commutation relations in the correlated ground state, Eq. (20),

$$\langle [\mathcal{A}_\mu, \mathcal{A}_{\mu'}^\dagger] \rangle = \delta_{\mu\mu'}, \quad \langle [\mathcal{R}_\lambda, \mathcal{R}_{\lambda'}^\dagger] \rangle = \delta_{\lambda\lambda'}, \tag{23}$$

if the amplitudes X and Y satisfy the same RPA orthogonality conditions (11), while the same RPA closure relations (12) guarantee the following inverse transformation of Eq. (18):

$$Q_p^\dagger = \sqrt{\langle D_p \rangle} \left[\sum_\mu X_p^{(\mu)} \mathcal{A}_\mu^\dagger + \sum_\lambda Y_p^{(\lambda)} \mathcal{R}_\lambda \right],$$

$$Q_h = \sqrt{\langle D_h \rangle} \left[\sum_\lambda X_h^{(\lambda)} \mathcal{R}_\lambda + \sum_\mu Y_h^{(\mu)} \mathcal{A}_\mu^\dagger \right]. \tag{24}$$

instead of Eq. (13).

The RRPA equations are obtained in the same way as that for the derivation of the RPA equations, assuming the factorization

$$\langle D_i D_j \rangle \simeq \langle D_i \rangle \langle D_j \rangle, \tag{25}$$

as well as neglecting all the expectation values $\langle Q_{p'}^\dagger Q_p \rangle$, $\langle Q_p Q_h \rangle$, and $\langle Q_h^\dagger Q_{h'} \rangle$. The RRPA submatrices obtained in this way for the addition modes have the form

$$A_{pp'}^{\mathrm{RRPA}} = 2 \left[\epsilon \left(p - \frac{1}{2} \right) + \frac{G}{2} \right] \delta_{pp'} - G \sqrt{\langle D_p \rangle \langle D_{p'} \rangle}, \tag{26}$$

$$B_{ph}^{\mathrm{RRPA}} = G \sqrt{\langle D_p \rangle \langle D_h \rangle}, \tag{27}$$

$$C_{hh'}^{\mathrm{RRPA}} = -2 \left[\epsilon \left(h - \frac{1}{2} \right) + \frac{G}{2} \right] \delta_{hh'} + G \sqrt{\langle D_h \rangle \langle D_{h'} \rangle}. \tag{28}$$

They are different from the RPA submatrices (15)–(17) by the square roots $\sqrt{\langle D_i \rangle \langle D_j \rangle}$, which renormalize the interaction. In the absence of GSC beyond RPA the expectation values $\langle D_i \rangle$ become 1, and the RPA submatrices are recovered.

The equations for the expectation values $\langle D_i \rangle$ are obtained following Refs. [4,6,7] in the form

$$\langle D_p \rangle = \frac{1}{1 + 2 \sum_\lambda \left[Y_p^{(\lambda)} \right]^2}, \quad \langle D_h \rangle = \frac{1}{1 + 2 \sum_\mu \left[Y_h^{(\mu)} \right]^2}, \tag{29}$$

which are exact in the present model because of the exact relation

$$N_i = 2 P_i^\dagger P_i. \tag{30}$$

For the details see Ref. [6]. The set of Eqs. (14), (26)–(28), and (29) forms the closed set of RRPA equations, which are non-linear due to the presence of the backward-going amplitudes Y in the expressions for the GSC factors $\langle D_i \rangle$ (29). In numerical calculations this set is solved self-consistently by

iteration. Knowing $\langle D_i \rangle$, one obtains the occupation numbers f_i from Eq. (21) as

$$f_p = \tfrac{1}{2}(1 - \langle D_p \rangle), \qquad f_h = \tfrac{1}{2}(1 + \langle D_h \rangle), \qquad (31)$$

so that within the RPA ($D_i = 1$) the HF occupation numbers $f_p = 0$ and $f_h = 1$ are recovered.

D. SCRPA

The only difference between the RRPA and SCRPA is that the latter includes the so-called screening factors, which are the expectations values $\langle Q_{p'}^\dagger Q_p \rangle$, $\langle Q_p Q_h \rangle$, and $\langle Q_h^\dagger Q_{h'} \rangle$ neglected within the RRPA. They are derived using the inverse transformation (24) and the definition of the correlated ground state (20), and have the form [6,7]

$$\langle Q_p^\dagger Q_{p'} \rangle = \langle P_p^\dagger P_{p'} \rangle = \sqrt{\langle D_p \rangle \langle D_{p'} \rangle} \sum_\lambda Y_p^{(\lambda)} Y_{p'}^{(\lambda)}, \qquad (32)$$

$$\begin{aligned} \langle Q_p Q_h \rangle = \langle Q_h^\dagger Q_p^\dagger \rangle &= -\langle P_h^\dagger P_p \rangle = -\langle Q_p^\dagger P_h \rangle \\ &= \sqrt{\langle D_p \rangle \langle D_h \rangle} \sum_\lambda X_h^{(\lambda)} Y_p^{(\lambda)}, \end{aligned} \qquad (33)$$

$$\langle Q_h^\dagger Q_{h'} \rangle = \sqrt{\langle D_h \rangle \langle D_{h'} \rangle} \sum_\mu Y_h^{(\mu)} Y_{h'}^{(\mu)} = \langle P_{h'}^\dagger P_h \rangle - \delta_{hh'} \langle D_h \rangle, \qquad (34)$$

with $\quad \langle P_{h'}^\dagger P_h \rangle = \sqrt{\langle D_h \rangle \langle D_{h'} \rangle} \sum_\lambda X_h^{(\lambda)} X_{h'}^{(\lambda)}.$

The SCRPA submatrices are then given as [6,7]

$$\begin{aligned} A_{pp'} = \langle [\overline{Q}_p, [H, \overline{Q}_{p'}^\dagger]] \rangle = 2 &\left\{ \left[\epsilon \left(p - \frac{1}{2} \right) + \frac{G}{2} \right] \right. \\ &+ \frac{G}{\langle D_p \rangle} \left[\sum_{p''} \langle Q_{p''}^\dagger Q_p \rangle - \sum_{h''} \langle Q_p Q_{h''} \rangle \right] \right\} \delta_{pp'} \\ &- G \frac{\langle D_p D_{p'} \rangle}{\sqrt{\langle D_p \rangle \langle D_{p'} \rangle}}, \end{aligned} \qquad (35)$$

$$B_{ph} = -\langle [\overline{Q}_p, [H, \overline{Q}_h]] \rangle = G \frac{\langle D_p D_h \rangle}{\sqrt{\langle D_p \rangle \langle D_h \rangle}}, \qquad (36)$$

$$\begin{aligned} C_{hh'} = -\langle [\overline{Q}_h, [H, \overline{Q}_{h'}^\dagger]] \rangle = -2 &\left\{ \left[\epsilon \left(h - \frac{1}{2} \right) + \frac{G}{2} \right] \right. \\ &+ \frac{G}{\langle D_h \rangle} \left[\sum_{h''} \langle Q_h^\dagger Q_{h''} \rangle - \sum_{p''} \langle Q_{p''}^\dagger Q_h^\dagger \rangle \right] \right\} \delta_{hh'} \\ &+ G \frac{\langle D_h D_{h'} \rangle}{\sqrt{\langle D_h \rangle \langle D_{h'} \rangle}}. \end{aligned} \qquad (37)$$

In principle no assumption on the factorization (25) is required within the SCRPA. However the factorization (25) does simplify greatly the calculations, whose results turn out to be quite close to those obtained without assuming Eq. (25) [6]. The equations for the GSC factors are the same as Eq. (29).

E. Two-particle separation energy and pair transition

1. Consistency in terms of two-particle separation energy

Unlike the particle-hole RPA, where the solutions with negative eigenvalues have no physical meaning, each set of the (particle-particle) RPA, RRPA, and SCRPA equations has $m + n$ solutions, all of which correspond to physical states. From them m ($= N/2$ in the present case) solutions correspond to the system with $N + 2$ particles, whose excitation energies related to the ground-state (g.s.) energy $\mathcal{E}_{\text{g.s.}}^{(N)}$ of the N-particle system are $E_\mu \equiv \mathcal{E}_\mu^{(N+2)} - \mathcal{E}_{\text{g.s.}}^{(N)}$ with $\mu = 1, \ldots, m$. The other n ($= N/2$) solutions have the eigenvalues equal to $-E_\lambda \equiv -[\mathcal{E}_{\text{g.s.}}^{(N)} - \mathcal{E}_\lambda^{(N-2)}]$ with $\lambda = 1, \ldots, n$. They correspond to the eigenstates of the system with $N - 2$ particles with the eigenvalues E_λ. Therefore the first eigenvalues $E_{\mu=1}$ of the $(N + 2)$-particle system and $E_{\lambda=1}$ of the $(N - 2)$-particle one are the ground-state energies of these systems related to that of the N-particle core. This means that they correspond to the two-particle separation energies $S_{2p}^{(+)}(N + 2)$ in the $(N + 2)$-particle system, and $S_{2p}^{(-)}(N)$ in the core, which are defined as

$$\begin{aligned} S_{2p}^{(+)}(N + 2) &= E_{\mu=1} \equiv \mathcal{E}_{\text{g.s.}}^{(N+2)} - \mathcal{E}_{\text{g.s.}}^{(N)}, \\ S_{2p}^{(-)}(N) &= E_{\lambda=1} \equiv \mathcal{E}_{\text{g.s.}}^{(N)} - \mathcal{E}_{\text{g.s.}}^{(N-2)}. \end{aligned} \qquad (38)$$

As such, in an exact or consistent theory, it follows from Eq. (38) that

$$S_{2p}^{(+)}(N + 2) = S_{2p}^{(-)}(N + 2). \qquad (39)$$

This equation is called the consistency condition hereafter, which means that the two ways of calculating the two-particle separation energy as the energy of the addition mode, which adds two particles to the N-particle core, and that of the removal mode, which removes two particles from the $(N + 2)$-particle system to get back to the same core, should give the same result. The expressions (38) were used in Ref. [8] to calculate the two-neutron separation energies for $N_c + 2$ and N_c nuclei, namely, ^{12,14}C, ^{10,12}Be, and ^{9,11}Li, whose neutron cores consist of $N_c = 8$ particles.

2. Pair-transition probability

The pair transition is generated by the operator

$$K = \sum_{i=p,h} (P_i^\dagger + P_i) = \sum_p (Q_p^\dagger + Q_p) - \sum_h (Q_h^\dagger + Q_h). \qquad (40)$$

By using the inverse transformation (24), one obtains the square $|\langle \widetilde{N}, 0 | \mathcal{A}_\mu K | \widetilde{N}, 0 \rangle|^2$ of the matrix element of the operator K, which corresponds to the transition probability from the ground state of the N-particle system to the one-phonon state $|\mu \rangle$ of the $(N + 2)$-particle system within the RRPA and/or SCRPA, in the following form:

$$\begin{aligned} B_\mu(N \to N + 2) &\equiv |\langle \widetilde{N}, 0 | \mathcal{A}_\mu K | \widetilde{N}, 0 \rangle|^2 \\ &= \left| \sum_p \sqrt{\langle D_p \rangle} X_p^{(\mu)} - \sum_h \sqrt{\langle D_h \rangle} Y_h^{(\mu)} \right|^2. \end{aligned} \qquad (41)$$

In the same way, the transition probability from the ground state of the N-particle system to the one-phonon state $|\lambda\rangle$ of the $(N-2)$-particle system is proportional to $|\langle \widetilde{N, 0}|\mathcal{R}_\lambda K|\widetilde{N, 0}\rangle|^2$, which is given as

$$B_\lambda(N-2 \leftarrow N) \equiv |\langle \widetilde{N, 0}|\mathcal{R}_\lambda K|\widetilde{N, 0}\rangle|^2$$

$$= \left| \sum_h \sqrt{\langle D_h \rangle} X_h^{(\lambda)} - \sum_p \sqrt{\langle D_p \rangle} Y_p^{(\lambda)} \right|^2. \quad (42)$$

The corresponding expressions within the RPA are recovered by setting $\langle D_i \rangle = 1$ in Eqs. (41) and (42). From the consistency condition discussed in the preceding section it follows that

$$B_{\mu=1}(N \to N+2) = B_{\lambda=1}(N \leftarrow N+2), \quad (43)$$

which means that the probabilities of the pair transition between the ground states of N- and $(N+2)$-particle systems should be the same in either direction. In the limit with $\epsilon_{p>1} = \epsilon_{p=1} = \epsilon/2$ and $\epsilon_{h>1} = \epsilon_{h=1} = -\epsilon/2$, the Richardson model is reduced to the well-known two-level model, in which each of the two levels is N-fold and the Fermi level (5) is located at $\epsilon_F = -G/2$. This degenerate model has been studied in detail in Refs. [13–15]. The RRPA solutions of this model yields the following two-particle separation energies [see, e.g, Eqs. (73) and (74) of Ref. [15]]:

$$S_{2p}^{(+)}(N+2) = \sqrt{(\epsilon + G)[\epsilon - G(\langle D \rangle N - 1)]},$$
$$S_{2p}^{(-)}(N+2) = \sqrt{(\epsilon + G)[\epsilon - G(\langle D \rangle N + 1)]}. \quad (44)$$

The RPA result is recovered by setting $\langle D \rangle = 1$. Result (44) analytically shows that, for this degenerate two-level model, the consistency condition (39) is asymptotically fulfilled only when $N \gg 1$.

III. ANALYSIS OF NUMERICAL CALCULATIONS

The calculations were carried out within the Richardson model with the level distance $\epsilon = 1$ MeV at several values of the particle number N as the pairing parameter G varies.

Shown in Fig. 1 are the two-particle separation energies $S_{2n}^{(\pm)}(N+2)$, which are obtained within the RPA, RRPA, and SCRPA according to Eq. (38) at $N = 2, 10, 50,$ and 100. For a comparison, the exact results, available for $N = 2$ and 10, are also shown. For $N = 50$ and 100 the size of the matrix makes the exact diagonalization infeasible so that the exact solutions are not available. In Ref. [6] it has been analyzed in detail that the RRPA and SCRPA extend the solutions far beyond the point where the RPA collapses, and that the SCRPA predictions are the closest ones to the exact solutions. Therefore the details of these features are not repeated here. This figure shows that, for $N \leqslant 10$, the consistency condition (39) is clearly violated in all the approximations under consideration. The violation is particular strong within the RPA at small N. It is seen that $S_{2n}^{(+)}(N+2) > S_{2n}^{(-)}(N+2)$ within the RPA and RRPA, but $S_{2n}^{(+)}(N+2) < S_{2n}^{(-)}(N+2)$ within the SCRPA. At a given value of G, the SCRPA predicts the smallest difference between $S_{2n}^{(+)}(N+2)$ and $S_{2n}^{(-)}(N+2)$, while the RPA gives the largest one. One can see how $S_{2n}^{(+)}(N+2)$
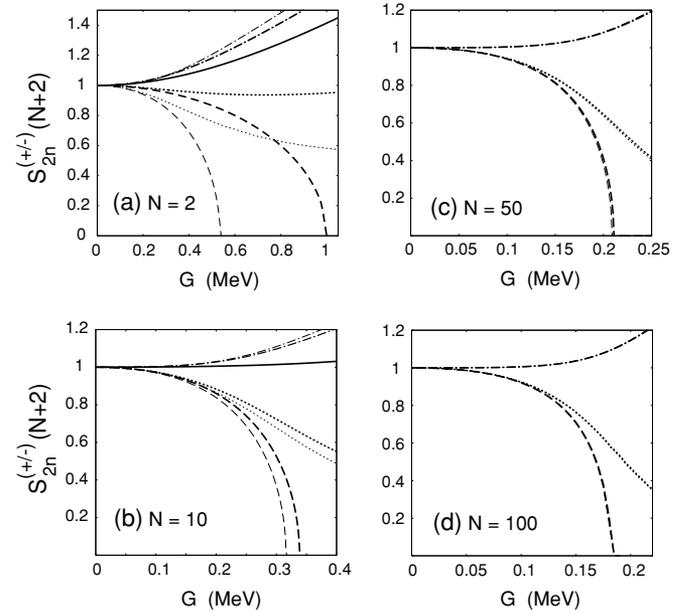

FIG. 1. Two-particle separation energies $S_{2n}^{(\pm)}(N+2)$ at several values of N as functions of pairing parameter G. The dashed, dotted, and dash-dotted lines represent results obtained within RPA, RRPA, and SCRPA, respectively, for which the thin lines denote $S_{2n}^{(-)}(N+2)$, while the thick lines stand for $S_{2n}^{(+)}(N+2)$. The exact results for $N = 2$ and 10 are shown by the thick solid lines.

gets closer to $S_{2n}^{(-)}(N+2)$ as N increases so that Eq. (39) is perfectly restored at $N = 100$. The small difference between $S_{2n}^{(+)}(N+2)$ and $S_{2n}^{(-)}(N+2)$ and the better agreement with the exact result, obtained within the SCRPA even at small N, show that the SCRPA is the most consistent approximation among three approximations under consideration. At the same time, the quite large difference between $S_{2n}^{(+)}(N+2)$ and $S_{2n}^{(-)}(N+2)$ obtained within the RPA at small N shows that the RPA is, in principle, an inconsistent theory when applied to light systems. This observation means that, GSC are important and cannot be neglected in calculations for light systems.

In order to see the origin of the effect due to GSC, the occupation numbers f_h of five hole levels, which are located nearest to the Fermi level, are plotted in Fig. 2 for $N = 10$ and 100. They are obtained by using the values of G equal to 0.33 MeV and 0.17 MeV for $N = 10$ and 100, respectively. These values are just below the collapsing points of the RPA, which are $G_{\text{crit}} = 0.34$ MeV and 0.18 MeV for $N = 10$ and 100, respectively. The results in Fig. 2(a) show that the QBA, on which the RPA is based, is no longer a good approximation when N is small. Indeed, the QBA means that $f_h = 1$, while the value of f_h, obtained for the level just below the Fermi level, is around 0.7 within the RPA. At large N, e.g. $N = 100$, the QBA is much better fulfilled as the occupation number $f_{h=1}$ is around 0.85, while those for all the other 49 hole levels are larger than 0.95 [Fig. 2(b)]. This figure also shows that the effect of GCS, once included within the RRPA and SCRPA, makes the RRPA and SCRPA phonon operators much closer to the ideal bosons since the values of f_h predicted by the RRPA and SCRPA for the hole level just below the
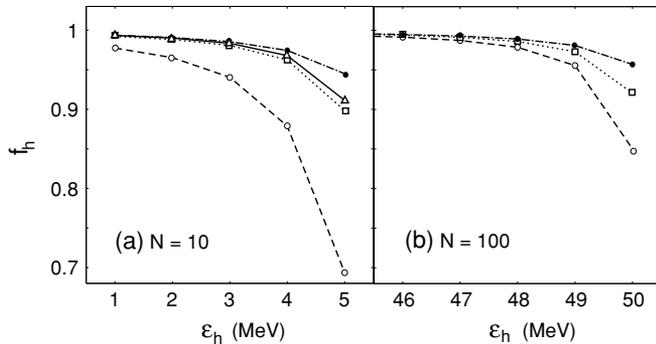
FIG. 2. Occupation numbers f_h of five hole levels nearest to the Fermi level ($h = 1, \ldots 5$) for the systems with $N = 10$ (a) and 100 (b). The open circles, squares, and full circles denote the RPA, RRPA, and SCRPA results, respectively. The triangles in (a) are the exact results. The dashed, dotted, dash-dotted, and solid lines connecting these discrete points are drawn to guide the eye. The results in (a) and (b) are obtained by using $G = 0.33$ MeV and 0.17 MeV, respectively.

Fermi level are around 0.9 and 0.95, respectively, for $N = 10$, and larger than these values at larger N. The comparison of the occupation numbers predicted by the RPA, RRPA, and SCRPA again shows that the QBA is best satisfied within the SCRPA. It is interesting to notice that, while the SCRPA predicts a two-particle separation energy much closer to the exact value for $N = 10$ as compared to the RPA [Fig. 1(b)], the agreement between the SCRPA prediction and exact values for the occupation number $f_{h=1}$ is not as good as that offered by the RRPA.

The pair-transition probabilities, shown in Fig. 3, increase with increasing G as one might expect. In a similar way as that for the two-particle separation energies, the condition (43) is
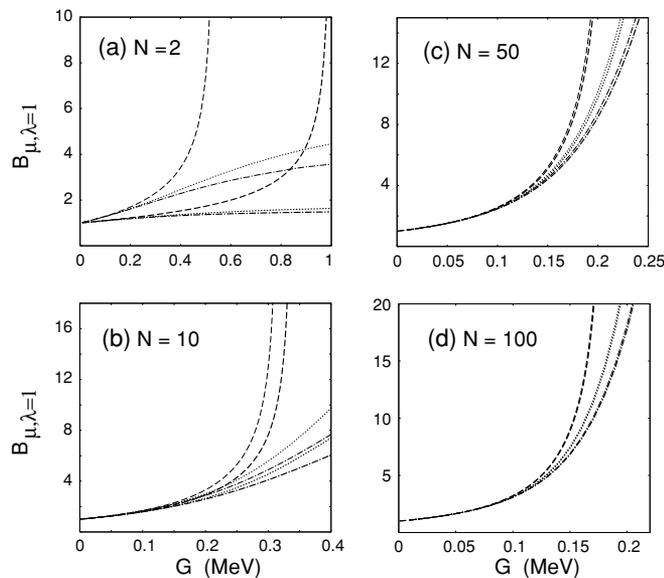

FIG. 3. Squares of matrix elements corresponding to the pair-transition probabilities as functions of pairing parameter G at several values of N. The notations for dashed, dotted, and dash-dotted lines are as in Fig. 1. The thick and thin lines denote $B_{\mu=1}(N \rightarrow N + 2)$ and $B_{\lambda=1}(N \leftarrow N + 2)$, respectively.

strongly violated within the RPA for small N. Combining the results on Figs. 1 and 3, it is seen that for $N \leqslant 10$, neither condition (39) nor (43) holds. Like the restoration of the consistency condition (39) for two-particle separation energies, as N increases the condition (43) for pair-transition probabilities is also gradually restored to become well-fulfilled at $N = 100$.

IV. CONCLUSIONS

The present paper tested the consistency of the RPA, RRPA, and SCRPA within the Richardson model. The consistency condition under consideration here is the requirement that the two-particle separation energy $S_{2n}^{(+)}(N + 2)$ obtained as the energy of the first addition mode, which adds two particles to the N-particle system, should be the same as $S_{2n}^{(-)}(N + 2)$ obtained as the energy of the first removal mode, which removes two particles from the $N + 2$-particle system. The results of calculations show that this consistency condition is well fulfilled only for sufficiently large values of the particle number N ($\geqslant 50$ in the present model), i.e., in medium-mass and heavy systems. For light systems ($N \leqslant 10$) it is found that, among the three approximations under consideration, the RPA, which has the largest deviation from the exact results at large G, strongly violates the consistency condition. The SCRPA agrees best with the exact solutions and also is the most consistent approximation in the sense that the above-mentioned consistency condition is well satisfied already at $N \geqslant 10$ even at G above the RPA collapsing point.

The results obtained in this test indicate that GSC beyond RPA becomes quite important in light systems, especially in the region close to the RPA collapsing point. They invalidate the QBA, which is assumed for the pair operators within the RPA. Therefore, in principle, the RPA predictions of the two-neutron separation energies in light neutron-rich nuclei cannot be reliable without taking the effect of GSC beyond RPA into account. From the three approximations considered in the present paper, the most consistent one, which includes this effect for light systems, turns out to be the SCRPA. Although the present test was conducted within a schematic model with doubly-fold equidistant levels, it is unlikely that this general conclusion on the inconsistency of the RPA would change when applied to realistic light nuclei. As a matter of fact, the recent calculations of the two-neutron separation energy S_{2n} in ^{12,14}Be by using the Gogny interaction have shown that the RRPA shifts the result by around 14% for ^{12}Be, and the consistency condition (39) is not fulfilled ($N = 10$) for both the RPA and RRPA [16]. An easy way to see if the RPA needs to be renormalized is to estimate the occupation numbers (31) using the backward-going amplitudes Y obtained within the RPA. If there is a strong deviation of f_p (f_h) from 0 (1), this means that the validity of the RPA is deteriorated. In the above-mentioned calculations for Be-isotopes e.g., the occupation number of the last occupied shell amounts to around 0.8 within the RPA, and increases to around 0.95 within the RRPA, showing the necessity of using the SCRPA, or at least the RRPA, instead of the RPA.

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