# Pairing correlations in finite Fermi systems

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# Summary and outline

- The reduced BCS model
  - Richardson formalism
  - evolution of the ground state with increasing pairing strength
- Reformulation of the ground state
  - definition of new collective pairs and their evolution
- Analysis in terms of elementary bosons
  - mapping procedure
  - analysis of the boson ground state
- Analogies with molecular BEC in ultracold Fermi gases

# The reduced BCS model

$$H = \sum_{j=1}^{\Omega} \epsilon_j N_j - g \sum_{i,j=1}^{\Omega} P_i^\dagger P_j$$

$$N_j = \sum_\sigma a_{j\sigma}^\dagger a_{j\sigma}, \quad P_j^\dagger = a_{j+}^\dagger a_{j-}^\dagger, \quad P_j = (P_j^\dagger)^\dagger$$

Assumptions:

- $\epsilon_j = jd$ (equally spaced levels)
- no partial occupation of the levels (only seniority-zero states)
- half-filling (number of levels ($\Omega$) = number of particles ($2N$))
- N even

# Exact solutions

$$H|\Psi\rangle = E^{(\Psi)}|\Psi\rangle$$

$$|\Psi\rangle = \prod_{i=1}^{N} B_i^\dagger |0\rangle, \qquad B_i^\dagger = \sum_{k=1}^{\Omega} \frac{1}{2\epsilon_k - E_i} P_k^\dagger$$

$$E^{(\Psi)} = \sum_{i=1}^{N} E_i$$

Richardson equations

$$1 - \sum_{k=1}^{\Omega} \frac{g}{2\epsilon_k - E_i} + \sum_{l(l\neq i)=1}^{N} \frac{2g}{E_l - E_i} = 0$$

# Exact solutions

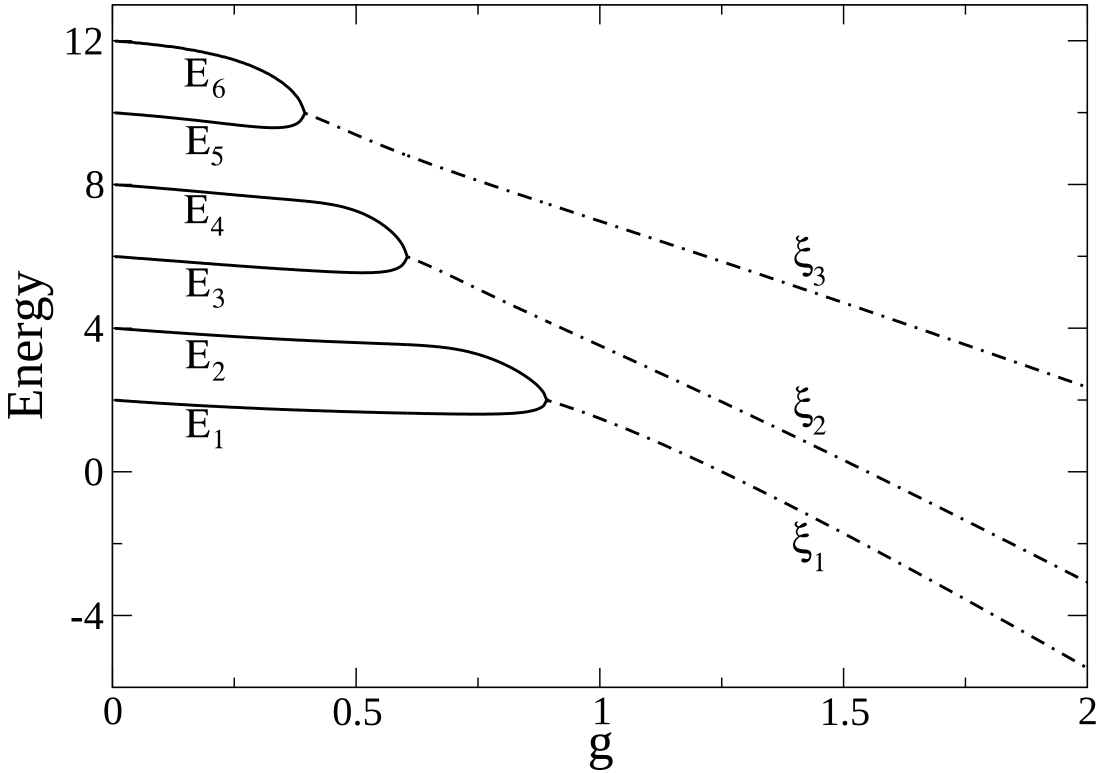
- Problem: singularity when $E_i = E_j$
- Solution: change of variables

$$E_{2\lambda-1} = \xi_\lambda - i\eta_\lambda, \quad E_{2\lambda} = \xi_\lambda + i\eta_\lambda \quad (\lambda = 1, 2, \cdots, N/2)$$

with $\xi_\lambda$ real and $\eta_\lambda$ either pure imaginary or real

- Result:
  - the Richardson equations depend only on the real quantities $\xi_\lambda$, $\eta_\lambda^2$
  - at the singularity $E_i$ and $E_j$ (real) turn into two complex-conjugate pair energies

# Pair energies $E_i$




$$|\Psi\rangle = \prod_{i=1}^{N} B_i^\dagger |0\rangle$$

$$B_i^\dagger = \sum_{k=1}^{\Omega} \frac{1}{2\epsilon_k - E_i} P_k^\dagger$$

$$(2N = \Omega = 12)$$

# Reformulating the ground state

$$|\Psi\rangle = \prod_{i=1}^{N} B_i^\dagger |0\rangle = \prod_{\lambda=1}^{N/2} B_{2\lambda-1}^\dagger B_{2\lambda}^\dagger |0\rangle$$

$$B_{2\lambda-1}^\dagger B_{2\lambda}^\dagger = (\Gamma_\lambda^\dagger)^2 + \eta_\lambda^2 (\Theta_\lambda^\dagger)^2$$

$$\Gamma_\lambda^\dagger = \sum_{k=1}^{\Omega} \frac{2\epsilon_k - \xi_\lambda}{(2\epsilon_k - \xi_\lambda)^2 + \eta_\lambda^2} P_k^\dagger, \qquad \Theta_\lambda^\dagger = \sum_{k=1}^{\Omega} \frac{1}{(2\epsilon_k - \xi_\lambda)^2 + \eta_\lambda^2} P_k^\dagger$$

N.B.: only real quantities are involved in these expressions

# Reformulating the ground state

Transformation:

$$\{\Gamma_\lambda^\dagger, \Theta_\lambda^\dagger\}_{\lambda=1,2,\cdots,N/2} \longrightarrow \{\Pi_\rho^\dagger\}_{\rho=1,2,\cdots,N}$$

The pairs $\Pi_\rho^\dagger$ result from the diagonalization of $H$ in the space

$$\{\Gamma_\lambda^\dagger|0\rangle, \Theta_\lambda^\dagger|0\rangle\}$$

They are such that

$$\langle 0|\Pi_\rho \Pi_{\rho'}^\dagger|0\rangle = \delta_{\rho\rho'}, \quad \langle 0|\Pi_\rho H \Pi_{\rho'}^\dagger|0\rangle = \tilde{\epsilon}_\rho \delta_{\rho\rho'}$$

# Reformulating the ground state

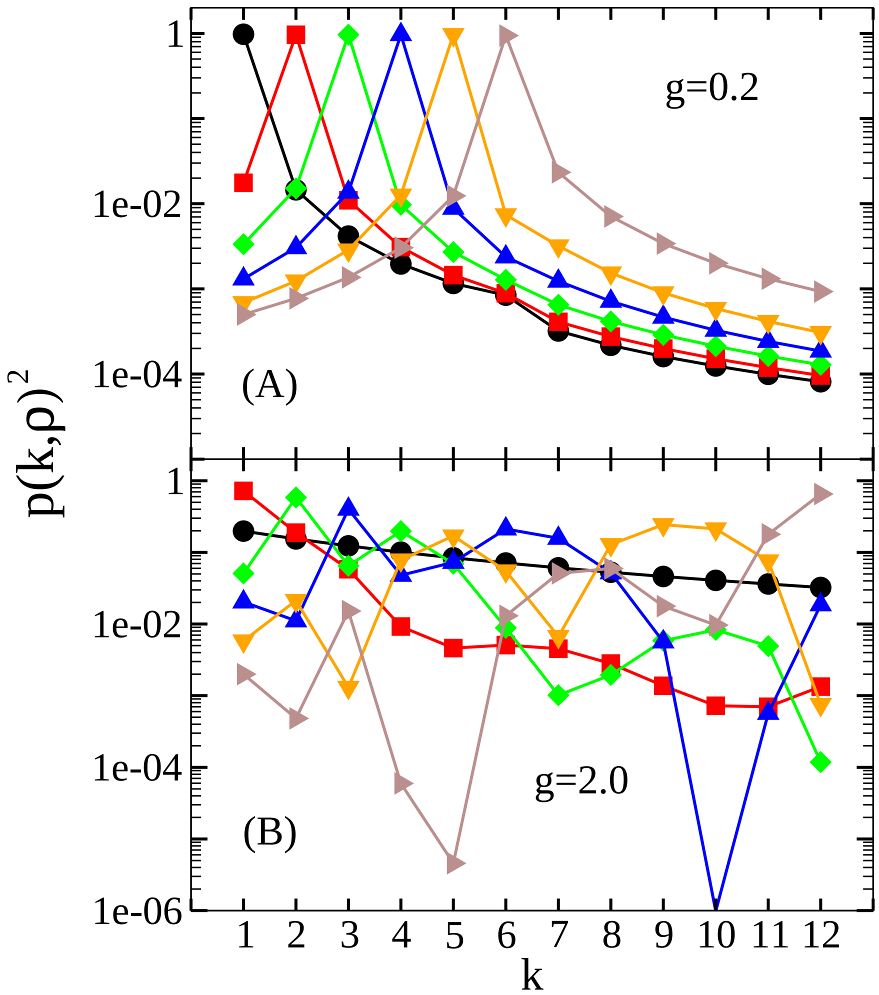
$$\Pi^\dagger_\rho = \sum_{\lambda=1}^{N/2} c(\lambda,\rho)\Gamma^\dagger_\lambda + \sum_{\lambda=1}^{N/2} d(\lambda,\rho)\Theta^\dagger_\lambda \equiv \sum_{k=1}^{2N} p(k,\rho)P^\dagger_k \quad (1 \le \rho \le N)$$

The exact ground state results from the diagonalization of $H$ in the space

$$F = \left\{ \Pi^\dagger_{\rho_1}\Pi^\dagger_{\rho_2}\cdots\Pi^\dagger_{\rho_N}|0\rangle \equiv |\rho\rangle \right\}_{1\le\rho_1\le\cdots\le\rho_N\le N}$$

N.B.:
F is not complete

# Occupation probabilities

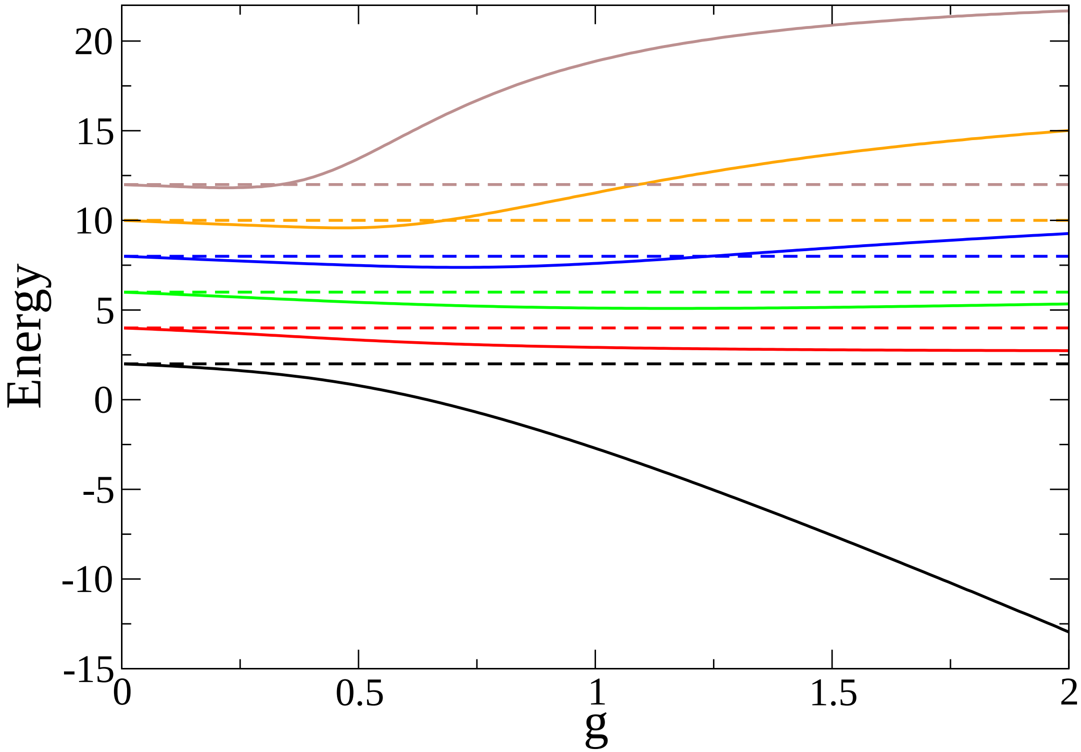



$$2N = \Omega = 12$$

$$\Pi_\rho^\dagger = \sum_{k=1}^{12} p(k,\rho) P_k^\dagger$$

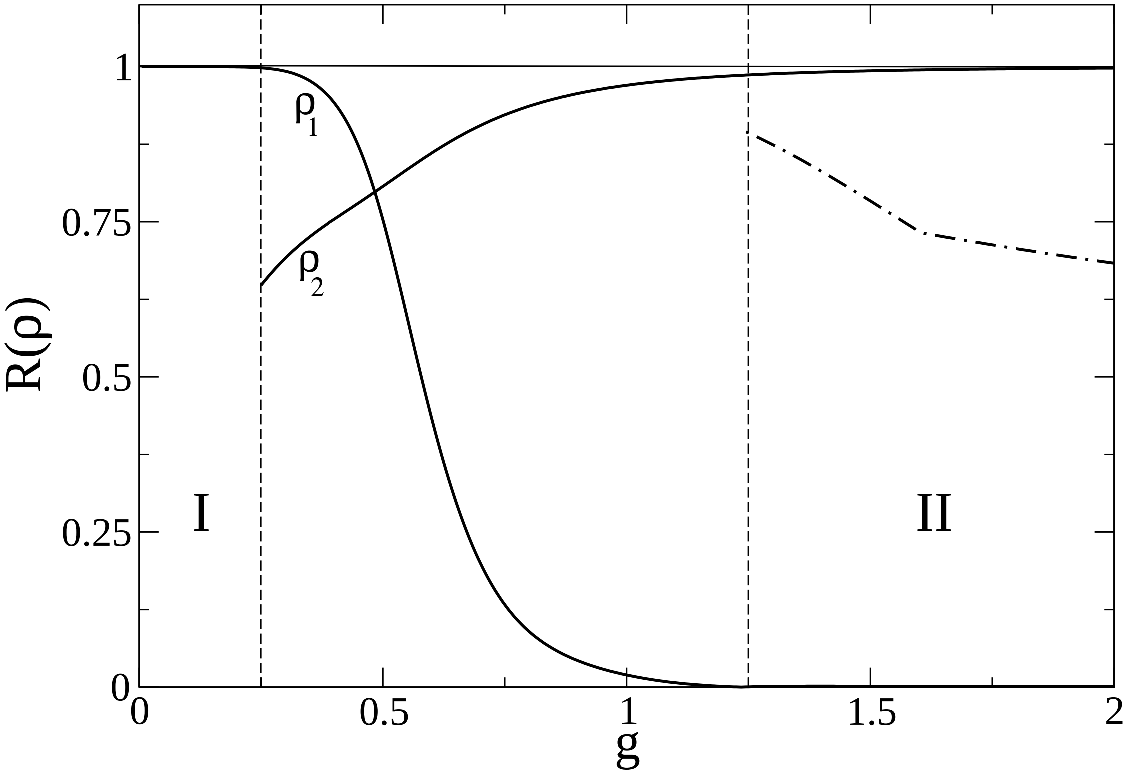
$$(1 \le \rho \le 6)$$

# Energies of the pairs $\Pi^\dagger_\rho$




$$\langle 0|\Pi_\rho H \Pi^\dagger_{\rho'}|0\rangle = \tilde{\epsilon}_\rho \delta_{\rho\rho'}$$

# Structure of the ground state

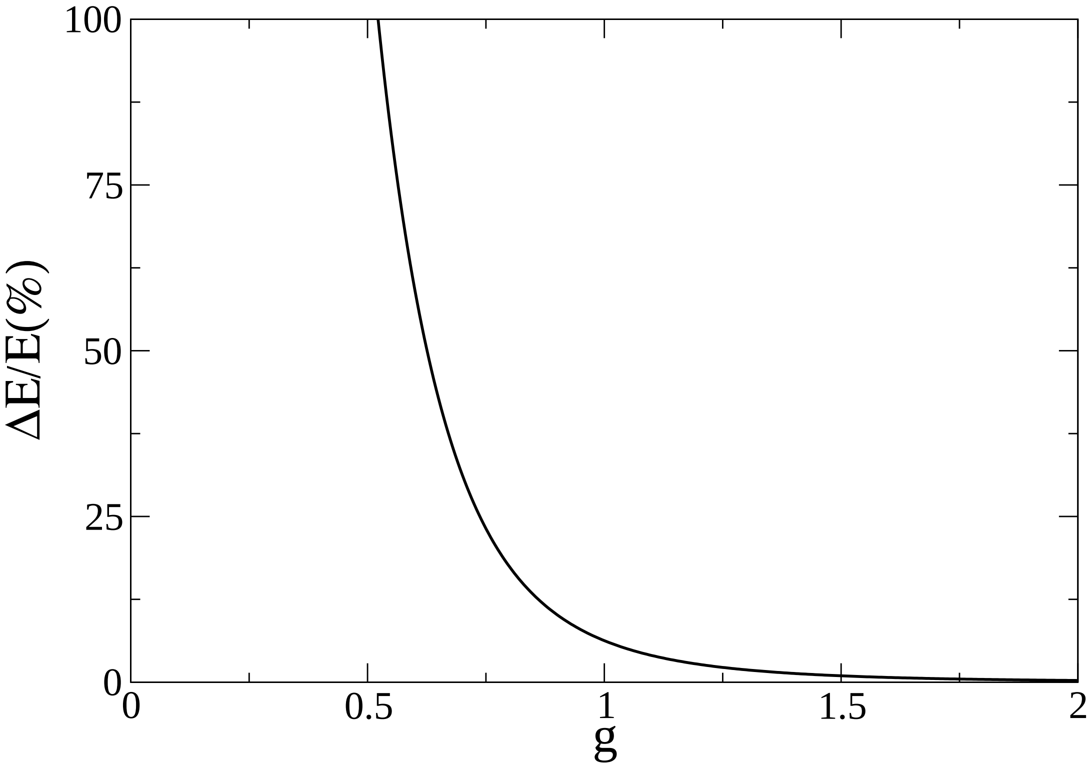



$$R(\rho) = \left| \frac{1}{\sqrt{\mathcal{N}_\rho}} \langle \Psi | \rho \rangle \right|$$

$$|\rho\rangle = \Pi^\dagger_{\rho_1} \Pi^\dagger_{\rho_2} \cdots \Pi^\dagger_{\rho_6} |0\rangle$$

$$|\rho_1\rangle = \Pi^\dagger_1 \Pi^\dagger_2 \Pi^\dagger_3 \Pi^\dagger_4 \Pi^\dagger_5 \Pi^\dagger_6 |0\rangle, \quad |\rho_2\rangle = (\Pi^\dagger_1)^6 |0\rangle$$

# Correlation energy




$$\frac{\Delta E}{E} = \frac{E - E'}{E}$$

$$E = \langle \Psi | H | \Psi \rangle - \langle HF | H | HF \rangle$$

$$E' = \frac{1}{\mathcal{N}_1} \langle 0 | (\Pi_1)^6 H (\Pi_1^\dagger)^6 | 0 \rangle - \langle HF | H | HF \rangle$$

# Mapping

- Fermion space

$$\Pi_k^\dagger$$

$$[\Pi_k^\dagger, \Pi_{k'}^\dagger] = 0, \qquad [\Pi_k, \Pi_{k'}^\dagger] = \delta_{kk'} + \cdots$$

$$F = \left\{ \Pi_{k_1}^\dagger \Pi_{k_2}^\dagger \cdots \Pi_{k_N}^\dagger |0\rangle \right\}_{1 \le k_1 \le \cdots \le k_N \le N}$$

- Boson space

$$b_k^\dagger$$

$$[b_k^\dagger, b_{k'}^\dagger] = 0, \qquad [b_k, b_{k'}^\dagger] = \delta_{kk'}$$

$$B = \left\{ b_{k_1}^\dagger b_{k_2}^\dagger \cdots b_{k_N}^\dagger |0) \right\}_{1 \le k_1 \le \cdots \le k_N \le N}$$

# Mapping

- Transformation operator

$$V \equiv |0)\langle 0| + \sum_{k_1} |\widetilde{1, k_1})\langle\widetilde{1, k_1}| + \sum_{k_2} |\widetilde{2, k_2})\langle\widetilde{2, k_2}| + \cdots$$
$$= \sum_{n,k_n} |\widetilde{n, k_n})\langle\widetilde{n, k_n}|$$

- Boson image of a fermion operator $T$

$$T_B \equiv VTV^\dagger = \sum_{n,k_n} \sum_{n',k_{n'}} |\widetilde{n, k_n})\langle\widetilde{n, k_n}|T|\widetilde{n', k_{n'}}\rangle(\widetilde{n', k_{n'}}|$$

- Projection operator

$$|0)(0| = 1 - \sum_k b_k^\dagger b_k + O(4)$$

- General property of $T_B$

$$(\widetilde{n, k}|T_B|\widetilde{n', k'}) = \langle\widetilde{n, k}|T|\widetilde{n', k'}\rangle$$

# Mapping (results)

- Boson Hamiltonian

$$H_B = \sum_i \tilde{\epsilon}_i b_i^\dagger b_i + V_B$$

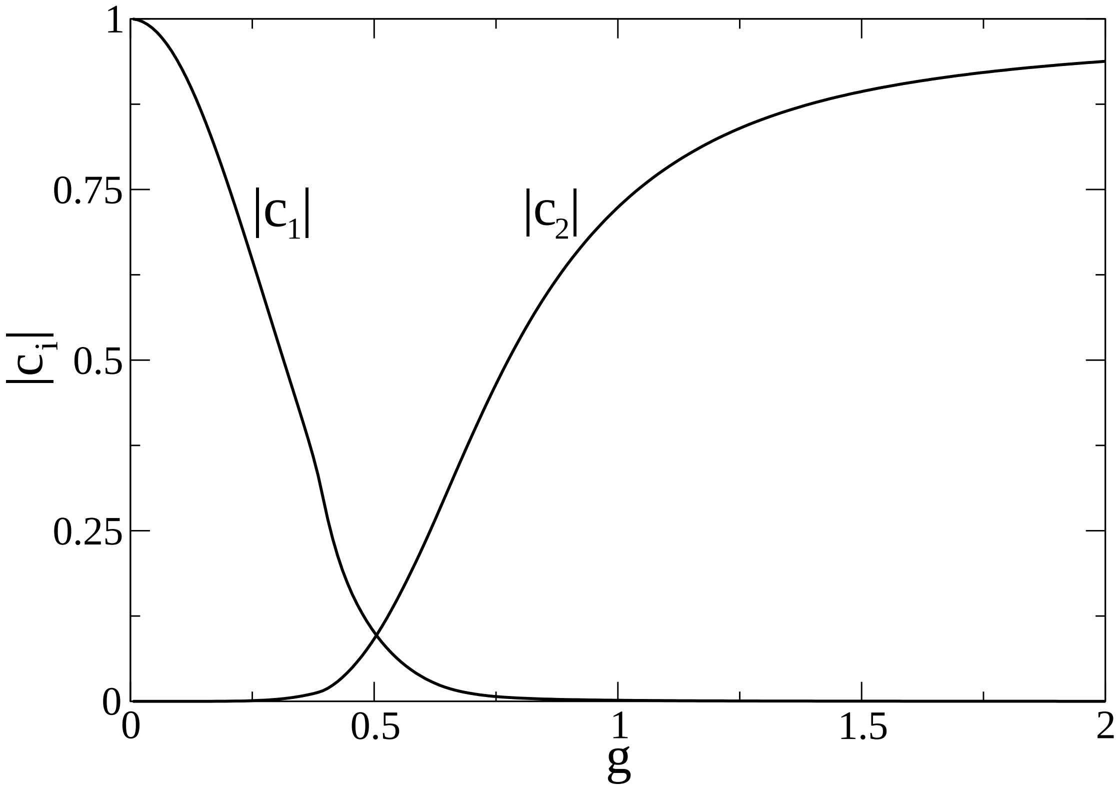
where:

  - $\tilde{\epsilon}_i$ is the energy of the pair $\Pi_i^\dagger$
  - $V_B$ is an interaction term which contains up to N-body boson operators

- Boson ground state

$$\begin{aligned} |\Psi) &= \sum_i c_i |i) \\ &= \sum_i c_i \frac{1}{\sqrt{\mathcal{N}_i}} b_{i_1}^\dagger b_{i_2}^\dagger \cdots b_{i_N}^\dagger |0) \end{aligned}$$

# Structure of the boson ground state

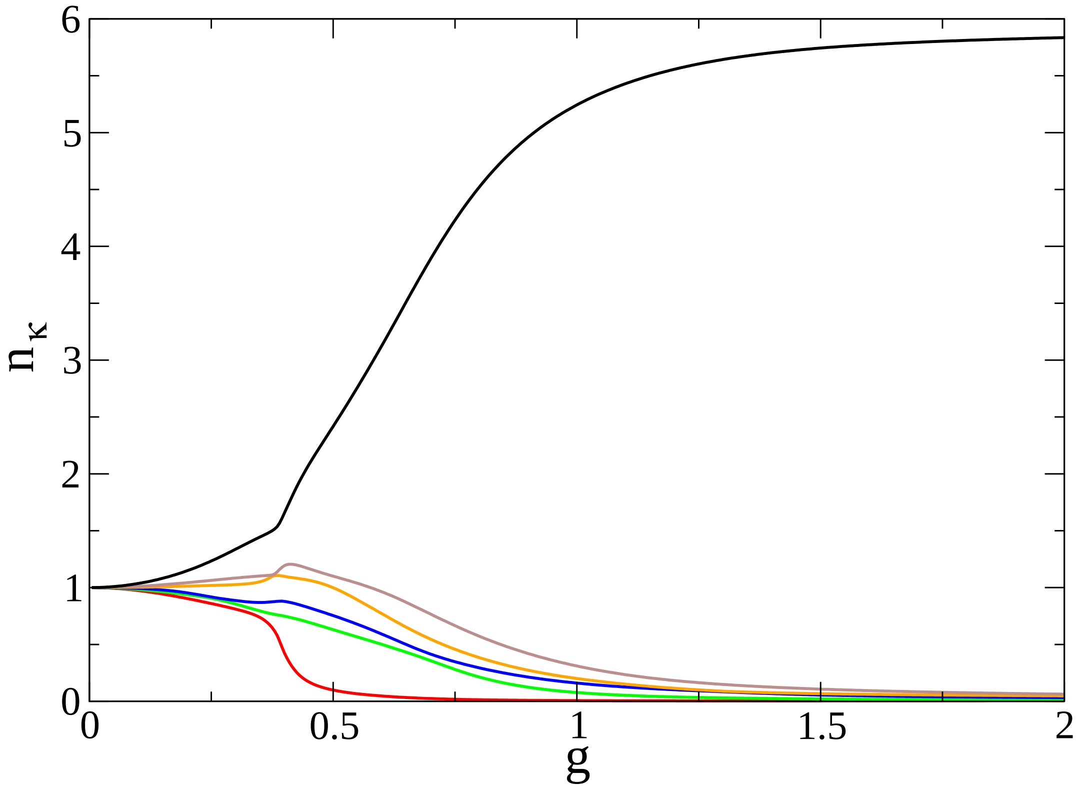



$$|\Psi) = \sum_i c_i |i)$$

$$(\sum_i c_i^2 = 1)$$

$$|1) = b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger b_5^\dagger b_6^\dagger |0), \qquad |2) = \frac{1}{\sqrt{\mathcal{N}}} (b_1^\dagger)^6 |0)$$

# One-body boson density




$$n(i,j) = (\Psi|b_i^\dagger b_j|\Psi)$$

$$(\Psi|\beta_k^\dagger \beta_{k'}|\Psi) = n_k \delta_{kk'} \qquad \beta_k^\dagger = \sum_i \phi_{ik} b_i^\dagger \qquad \left(\sum_k n_k = N\right)$$

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